Off-center moments set of Vlasov-Maxwell system

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We propose a more effective fluid description on Vlasov-Maxwell (V-M) system. It is via an open set of off-center moments, which obey an open set of motion equations. This new description is of more advantage to give exact macroscopic information of the V-M system than the well-known moments-description. The new description implies that obtaining exact solutions of all moments is not necessary condition of obtaining those of self-consistent fields.

Fluid description on a Vlasov-Maxwell (V-M) system [1] is often via an open set of moments $\{M_i, 0 =< i < \infty\}$, where $M_i = \int v^i f d^3v$ and those M_i obey an open set of fluid equations $\left\{\int v^i \widehat{L} f d^3v = 0, 0 =< i < \infty\right\}$. Here, $\widehat{L} f = 0$ is Vlasov equation (VE) [2] and the operator is $\widehat{L} = [\partial_t + v \cdot \nabla - LF_v \cdot \partial_p]$, where $LF_v = e\left[E(r,t) + v \times B(r,t)\right]$ represents the Lorentz force, $p(v) = v\Gamma(v)$ and $\Gamma(v) = \frac{1}{\sqrt{1-v \cdot v}}$. The open equation set $\left\{\int v^i \widehat{L} f d^3v = 0, 0 =< i < \infty\right\}$ reflect relations among all moments. On the other hand, Maxwell equations (MEs) reflect a relation between self-consistent fields (E, B) and (M_0, M_1). This seems to suggest that (E, B) will be dependent on all moments and hence is nearly impossible to be exactly solved (because there will be dependence relations, in infinite-number, of (E, B) on each moment to be calculated. This raises naturally a fundamental question, whether exact solutions of all moments are necessary for obtaining exact solutions of (E, B)?

We can define another open set $\{D_i, 0 =< i < \infty\}$, where $D_i = \left[\frac{M_i}{M_0} - \left(\frac{M_1}{M_0}\right)^i\right] * M_0$ through the M-set. Clearly, $D_0 = 0$ and $D_1 = 0$ automatically exist. According to MEs, (E, B) depend on (M_0, M_1) and is independent of the D-set. But the D-set is governed by (E, B) through the open equation set $\left\{\int v^i \widehat{L} f d^3v = 0, 1 =< i < \infty\right\}$, where each equation $\int v^i \widehat{L} f d^3v = 0$ can be expressed through the D-set

$$A_i \partial_t D_i + B_{i+1} \nabla D_{i+1} + \sum_{m>=i+1} C_m D_m = 1, \tag{1}$$

and coefficients A_i, B_i, C_i are known functionals of E, B, M_0, M_1. Starting from the $i = 1$

case, we can formally obtain an expression of D_2 in all terms $D_{i>=3}$, and then substituting it into the $i=2$ case and formally obtain an expression of D_3 in all terms $D_{i>=4}$, ... Finally, we will find that all $D_{i>=2}$ are determined by D_∞ and all coefficients A_i, B_i, C_i. Namely, the open equation set $\left\{\int v^i \widehat{L} f d^3 v = 0, 1 =< i < \infty\right\}$ does not lead to a substantial constraint on E, B, M_0, M_1.

The introduction of the D-set resolves a realistic obstacle in obtaining exact solutions of (E, B). Although (E, B) is affected by each M_i, it is only a (M_0, M_1)-dependent part of each M_i that has a substantial contribution to (E, B). Each M_i has a part $D_i * M_0$ having no contribution to (E, B). The open equation set $\left\{\int v^i \widehat{L} f d^3 v = 0, 1 =< i < \infty\right\}$ reveals relations among those D_i. In short, exact solutions of the D-set is not a necessary condition for those of (E, B).

For any V-M system, there is following theorem.

Theorem: For Vlasov equation (VE) $\widehat{L} f = 0$, there will be 1). $\left[\widehat{L} + \left(v \cdot \nabla \frac{\int F v d^3 v}{\int F d^3 v}\right) \partial_v\right] F = 0$, where $F(f) \equiv \delta(v-u) * \int [f * \delta(v-u)] d^3 v$ and $u = \frac{\int f v d^3 v}{\int f d^3 v}$; 2). $\partial_t n_0 + u \cdot \nabla n_0 = 0$, where $n_0 = \int [f * \delta(v-u)] d^3 v < M_0$ (because $f >= 0$ always exists); 3). $\partial_t \frac{u}{\sqrt{1-u^2}} + e[E + u \times B] = 0$.

Proof: According to above definitions, we note that there naturally exists $u = \frac{\int f v d^3 v}{\int f d^3 v} = \frac{\int F v d^3 v}{\int F d^3 v}$ and also have

$$\begin{aligned} \widehat{L} F &= [\partial_t + v \cdot \nabla] n_0 * \delta(v-u) + n_0 * [\partial_t + v \cdot \nabla - LF_v \cdot \partial_p] \delta(v-u) \\ &= [\partial_t + u \cdot \nabla] n_0 * \delta - n_0 * [\partial_t u + v \cdot \nabla u + LF_v \cdot \partial_p v] * \delta', \end{aligned} \tag{2}$$

where we have utilized properties of Dirac function: $x\delta(x) = 0$ and $x\delta'(x) = -\delta(x)$. Shifting $n_0 * v \cdot \nabla u * \delta'$, which is just $v \cdot \nabla u \partial_v F$ from righthand side to lefthand, we have

$$\begin{aligned} \widehat{L} F + v \cdot \nabla u \partial_v F &= [\partial_t + u \cdot \nabla] n_0 * \delta - n_0 * [\partial_t u + LF_v \cdot \partial_p v] * \delta' \\ &= [\partial_t + u \cdot \nabla] n_0 * \delta - n_0 * [\partial_t u + (LF_v \cdot \partial_p v)|_{v=u}] * \delta', \end{aligned} \tag{3}$$

where we also have utilized $x\delta(x) = 0$ and $x\delta'(x) = -\delta(x)$. For simplicity of symbols, we denote $\widehat{L} F + v \cdot \nabla u \partial_v F$ as Ω, and then it is easy to verify following relations

$$[\partial_t + u \cdot \nabla] n_0 = \int \Omega d^3 v \tag{4}$$

$$n_0 * [\partial_t u + (LF_v \cdot \partial_p v)|_{v=u}] = -\int [v-u] * \Omega d^3 v. \tag{5}$$

. Clearly, Eq.(2), or $0 = \Omega - \int \Omega d^3\upsilon * \delta + \int [\upsilon - u] * \Omega d^3\upsilon * \delta'$, leads to 1)$0 = \Omega$ 2) $0 = \int \Omega d^3\upsilon = [\partial_t + u \cdot \nabla] n_0$ and 3).$0 = [\partial_t u + (LF_\upsilon \cdot \partial_p \upsilon) |_{\upsilon=u}]$ or $\partial_t \frac{u}{\sqrt{1-u^2}} + e [E + u \times B] = 0$. The theorem is thus strictly proven.

[1] N. A. Krall and A. W. Trivelpiece, New York:*Principles of plasma physics*, McGraw-Hill, 1977.

[2] A. Vlasov, J. Phys. U.S.S.R **10**,25(1945).